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Cesàro Summability of Two-Parameter Walsh–Fourier Series

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A new atomic decomposition of the two-parameter dyadic martingale Hardy spaces H_p defined by the quadratic variation is given. We introduce H_p-quasi-local operators and prove that if a sublinear operator V is H_p-quasi-local and bounded from L_2 to L_2 then it is also bounded from H_p to L_p $(0<p\leqslant 1)$. By an interpolation theorem we get that V is of weak type $(H_1^{\#}, L_1)$ where the Hardy space $H_1^{\#}$ is defined by the hybrid maximal function. As an application it is shown that the maximal operator of the Cesàro means of a two-parameter martingale is bounded from H_p to L_p $(4/5<p\leqslant\infty)$ and is of weak type $(H_1^{\#}, L_1)$. So we obtain that the Cesàro means of a function $f\in H_1^{\#}$ converge a.e. to the function in question. Finally, it is verified that if the supremum is taken over all two-powers, only, then the maximal operator of the Cesàro means is bounded from H_p to L_p for every $2/3<p\leqslant\infty$. © 1997 Academic Press

1. INTRODUCTION

For double trigonometric Fourier series Marcinkievicz and Zygmund [16] proved that the Cesàro means $\sigma_{n,m}f$ of a function $f\in L_1$ converge a.e. to f as $n, m\to\infty$ provided that the pairs (n, m) are in a positive cone, i.e., provided that $m/n\leqslant\alpha$ and $n/m\leqslant\alpha$. This result for double Walsh–Fourier series is verified by the author [25].

It is known that, for double Walsh–Fourier series, $\sigma_{n,m}f\to f$ in L_p norm as $\min(n, m)\to\infty$ whenever $f\in L_p$ for some $1\leqslant p<\infty$. Moreover, if $1<p<\infty$ then the convergence holds a.e., too (see Weisz [28]). Móricz *et al.* [18] have proved that if $f\in L\log L$ then the Cesàro summability holds.

The Hardy–Lorentz spaces $H_{p,q}^{\square}$ and $H_{p,q}$ of two-parameter martingales on the unit square are defined by the $L_{p,q}$ Lorentz norms of the diagonal maximal function $\sup_{n\in\mathbf{N}}|f_{n,n}|$ and of the two-parameter quadratic variation $(0<p, q\leqslant\infty)$, respectively. Of course, $H_p^{\square}=H_{p,p}^{\square}$ and $H_p=H_{p,p}$ are the usual Hardy spaces $(0<p\leqslant\infty)$. Note that $H_{p,q}\subset H_{p,q}^{\square}$.

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The following maximal operators of the Cesàro means are to be investigated: $\sigma^* f$, resp. σf, is defined by the supremum over $\mathbf{N}^2$ of $|\sigma_{n,m} f|$, resp. $|\sigma_{2^n, 2^m} f|$. Let $\sigma^\alpha f$ be the supremum over a positive cone of $|\sigma_{n,m} f|$. In the one-dimensional case it is known that σ^* is bounded from H_1 to L_1 and is of weak type (L_1, L_1), i.e.,

$$\sup_{\alpha > 0} \alpha \lambda(\sigma^* f > \alpha) \leqslant C \|f\|_1$$

whenever $f \in L_1$ (see Fujii [12] and Schipp [21]). It was proved by Móricz *et al.* [18] that the operator σ is of weak type $(H_1^\#, L_1)$ where $H_1^\#$ is defined by the expectation of the hybrid maximal function $\sup_{n \in \mathbf{N}} |f_{n,\infty}|$. Moreover, σ^* is bounded from L_p to L_p $(1 < p < \infty)$ and σ^α is bounded from $H_{p,q}^\square$ to $L_{p,q}$ for $1/2 < p < \infty$ and $0 < q \leqslant \infty$ and is of weak type (L_1, L_1) (see Weisz [28] and [25]).

In this paper we extend these results. A new atomic decomposition of H_p is given; more exactly, the H_p-atoms are decomposed into the sum of "elementary (rectangle) particles." By this theorem, in the definition of the H_p-quasi-local operators it is enough to take rectangle H_p-atoms. An operator V is H_p-quasi-local $(0 < p \leqslant 1)$ if there exists $\delta > 0$ such that for every rectangle H_p-atom a and for every $r \geqslant 1$ the integral of $|Va|^p$ over $[0, 1)^2 \backslash R^r$ is less than $C_p 2^{-\delta r}$ where the dyadic rectangle R is the support of a and R^r is the 2^r-fold dilation of R. With the help of Journé's covering lemma [15] we show that a sublinear and H_p-quasi-local operator V which is bounded from L_2 to L_2 is also bounded from H_p to L_p $(0 < p \leqslant 1)$. We get with interpolation that V is bounded from $H_{p,q}$ to $L_{p,q}$ $(0 < p < 2$, $0 < q \leqslant \infty)$ as well and is of weak type $(H_1^\#, L_1)$. The analogous results for the classical Hardy space are due to Chang and Fefferman [7, 8].

It will be shown that σ^* is H_p-quasi-local for each $4/5 < p \leqslant 1$. Consequently, σ^* is bounded from $H_{p,q}$ to $L_{p,q}$ for $4/5 < p < \infty$, $0 < q \leqslant \infty$, and is of weak type $(H_1^\#, L_1)$. A usual density argument implies that $\sigma_{n,m} f \to f$ a.e. as $\min(n, m) \to \infty$ whenever $f \in H_1^\#$. Finally, it is proved that the operator σ is H_p-quasi-local for each $2/3 < p \leqslant 1$ and so, by interpolation, it is bounded from $H_{p,q}$ to $L_{p,q}$ for every $2/3 < p < \infty$ and $0 < q \leqslant \infty$.

2. MARTINGALES AND HARDY SPACES

For a set $\mathbf{X} \neq \emptyset$ let $\mathbf{X}^2$ be its Cartesian product $\mathbf{X} \times \mathbf{X}$ taken with itself. An element from $\mathbf{N}^2$ will be denoted by (n, m) or simple by n. In this paper the unit square $[0, 1)^2$ and the two-dimensional Lebesgue measure λ are to be considered.

By a *dyadic interval* we mean one of the form $[k2^{-n}, (k+1)\,2^{-n})$ for some $k, n \in \mathbf{N}$, $0 \leqslant k < 2^n$. Given $n \in \mathbf{N}$ and $x \in [0, 1)$ let $I_n(x)$ denote the dyadic interval of length 2^{-n} which contains x. The Cartesian product of two dyadic intervals is said to be a *dyadic rectangle*. Clearly, the dyadic rectangle of area $2^{-n} \times 2^{-m}$ containing $(x, y) \in [0, 1)^2$ is given by

$$I_{n,m}(x, y) := I_n(x) \times I_m(y).$$

The σ-algebra generated by the dyadic rectangles $\{I_{n,m}(x): x \in [0, 1)^2\}$ will be denoted by $\mathcal{F}_{n,m}$ $(n, m \in \mathbf{N})$, more precisely,

$$\mathcal{F}_{n,m} = \sigma\{[k2^{-n}, (k+1)\,2^{-n}) \times [l2^{-m}, (l+1)\,2^{-m}) : 0 \leqslant k < 2^n, 0 \leqslant l < 2^m\}$$

where $\sigma(\mathcal{H})$ denotes the σ-algebra generated by an arbitrary set system $\mathcal{H}$.

Introduce the following σ-algebras:

$$\mathcal{F}_{n_1,\infty} := \sigma\left(\bigcup_{k=0}^{\infty} \mathcal{F}_{n_1,k}\right), \mathcal{F}_{\infty,n_2} := \sigma\left(\bigcup_{k=0}^{\infty} \mathcal{F}_{k,n_2}\right) \quad (n = (n_1, n_2) \in \mathbf{N}^2).$$

The expectation and the conditional expectation operators relative to $\mathcal{F}_n$, $\mathcal{F}_{n_1,\infty}$, and $\mathcal{F}_{\infty,n_2}$ $(n \in \mathbf{N}^2)$ are denoted by E, E_n, $E_{n_1,\infty}$, and E_{∞,n_2}, respectively. We briefly write L_p or $L_p[0, 1)^2$ instead of the real $L_p([0, 1)^2, \lambda)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p := (E\,|f|^p)^{1/p}$ $(0 < p \leqslant \infty)$. For simplicity, we assume that for a function $f \in L_1$ we have $E_{n,0} f = E_{0,n} f = 0$ $(n \in \mathbf{N})$.

An integrable sequence $f = (f_n, n \in \mathbf{N}^2)$ is said to be a *martingale* if

(i) it is *adapted*, i.e., f_n is $\mathcal{F}_n$ measurable for all $n \in \mathbf{N}^2$

(ii) $E_n f_m = f_n$ for all $n \leqslant m$, where for $n = (n_1, n_2)$, $m = (m_1, m_2) \in \mathbf{N}^2$, $n \leqslant m$ means that $n_1 \leqslant m_1$ and $n_2 \leqslant m_2$.

For simplicity, we always suppose that for a martingale f we have $f_n = 0$ if $n_1 = 0$ or $n_2 = 0$. Of course, the theorems that are to be proved later are true with a slight modification without this condition, too.

The martingale $f = (f_n, n \in \mathbf{N}^2)$ is said to be *L_p-bounded* $(0 < p \leqslant \infty)$ if $f_n \in L_p$ $(n \in \mathbf{N}^2)$ and

$$\|f\|_p := \sup_{n \in \mathbf{N}^2} \|f_n\|_p < \infty.$$

If $f \in L_1$ then it is easy to show that the sequence $\tilde{f} = (E_n f, n \in \mathbf{N}^2)$ is a martingale. Moreover, if $1 \leqslant p < \infty$ and $f \in L_p$ then $\tilde{f}$ is L_p-bounded and

$$\lim_{\min(n_1, n_2) \to \infty} \|E_n f - f\|_p = 0,$$

consequently,

$$\|\tilde{f}\|_p = \|f\|_p$$

(see Neveu [19]). The converse of the latest proposition holds also if $1 < p < \infty$ (see Neveu [19]): for an arbitrary martingale $f = (f_n, n \in \mathbf{N}^2)$ there exists a function $g \in L_p$ for which $f_n = E_n g$ if and only if f is L_p-bounded. If $p = 1$ then there exists a function $g \in L_1$ of the preceeding type if and only if f is *uniformly integrable* (see Neveu [19]), namely, if

$$\lim_{y \to \infty} \sup_{n \in \mathbf{N}^2} \int_{\{|f_n| > y\}} |f_n| \, dP = 0.$$

Thus the map $f \mapsto \tilde{f} := (E_n f, n \in \mathbf{N}^2)$ is isometric from L_p onto the space of L_p-bounded martingales when $1 < p < \infty$. Consequently, these two spaces can be identified with each other. Similarly, the L_1 space can be identified with the space of uniformly integrable martingales. For this reason a function $f \in L_1$ and the corresponding martingale $(E_n f, n \in \mathbf{N}^2)$ will be denoted by the same symbol f.

The distribution function of a Borel measurable function f is defined by

$$\lambda(\{|f| > \alpha\}) := \lambda(\{x : |f(x)| > \alpha\}) \qquad (\alpha \geqslant 0).$$

The *weak* L_p space L_p^* $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{L_p^*} := \sup_{\alpha > 0} \alpha \lambda(\{|f| > \alpha\})^{1/p} < \infty$$

while we set $L_\infty^* = L_\infty$.

The spaces L_p^* are special cases of the more general Lorentz spaces $L_{p,q}$. In their definition another concept is used. For a measurable function f the *non-increasing rearangement* is defined by

$$\tilde{f}(t) := \inf\{\alpha : \lambda(\{|f| > \alpha\}) \leqslant t\}.$$

Lorentz space $L_{p,q}$ is defined as follows: for $0 < p < \infty$, $0 < q < \infty$,

$$\|f\|_{p,q} := \left(\int_0^\infty \tilde{f}(t)^q \, t^{q/p} \frac{dt}{t} \right)^{1/q}$$

while for $0 < p \leqslant \infty$

$$\|f\|_{p,\infty} := \sup_{t > 0} t^{1/p} \tilde{f}(t).$$

Let

$$L_{p,q} := L_{p,q}([0,1)^2, \lambda) := \{f : \|f\|_{p,q} < \infty\}.$$

One can show the following equalities:

$$L_{p,p} = L_p, \qquad L_{p,\infty} = L_p^* \qquad (0 < p \leqslant \infty)$$

(see e.g. Bennett and Sharpley [1] or Bergh and Löfström [2]).

The *maximal function* and *hybrid maximal function* of a martingale $f = (f_{n,m};\, n, m \in \mathbf{N})$ are defined by

$$f^* := \sup_{n, m \in \mathbf{N}} |f_{n,m}|, \qquad f^{\#} := \sup_{n \in \mathbf{N}} |f_{n,\infty}|.$$

It is easy to see that, in the case $f \in L_1$, the maximal functions can also be given by

$$f^*(x, y) = \sup_{n, m \in \mathbf{N}} \frac{1}{\lambda(I_{n,m}(x, y))} \left| \int_{I_{n,m}(x,y)} f \, d\lambda \right|$$

and

$$f^{\#}(x, y) = \sup_{n \in \mathbf{N}} \frac{1}{\lambda(I_n(x))} \left| \int_{I_n(x)} f(t, y) \, dt \right|,$$

respectively.

We define the *martingale differences* by

$$d_n f := f_{n_1, n_2} - f_{n_1 - 1, n_2} - f_{n_1, n_2 - 1} + f_{n_1 - 1, n_2 - 1} \qquad (n \in \mathbf{N}^2)$$

and $d_{k,0} f = f_{0,k} f = 0$ $(k \in \mathbf{N})$.

It is easy to show that $(d_n f,\ n \in \mathbf{N}^2)$ is an integrable and adapted sequence. Moreover, one can conclude that

$$E_n \, d_m f = 0 \qquad (n \not\geqslant m). \tag{1}$$

Conversely, if an integrable and adapted function sequence $(d_n,\ n \in \mathbf{N}^2)$ has the property (1) then $(f_n,\ n \in \mathbf{N}^2)$ is a martingale where $f_n := \sum_{m \leqslant n} d_m$.

The *quadratic variation* of a martingale f is introduced with

$$S(f) := \left(\sum_{n \in \mathbf{N}^2} |d_n f|^2 \right)^{1/2}.$$

It was proved by Brossard [4, 5] and Metraux [17] that

$$\|S(f)\|_p \sim \|f^*\|_p \qquad (0<p<\infty) \tag{2}$$

where $\sim$ denotes the equivalence of the norms. The equivalences

$$\|f^*\|_p \sim \|f^{\#}\|_p \sim \|f\|_p \qquad (1<p\leqslant\infty) \tag{3}$$

follow from Doob's inequality (see Neveu [19], Cairoli [6]). For an arbitrary function $f \in L_1$ we have

$$\sup_{\alpha>0} \alpha\lambda(f^*>\alpha) \leqslant \|f^{\#}\|_1 \tag{4}$$

and

$$\sup_{\alpha>0} \alpha\lambda(S(f)>\alpha) \leqslant C\,\|f^{\#}\|_1. \tag{5}$$

Note that (4) was proved by Weisz [27] and (5) by Frangos and Imkeller [11]. On the right-hand sides of (4) and (5), $\|f^{\#}\|_1$ cannot be replaced by $\|f\|_1$; counterexamples can be found for the first case in Cairoli [6] and for the second case in Imkeller [14].

For $0<p, q\leqslant\infty$ the *martingale Hardy–Lorentz spaces* $H_{p,q}$ and $H^{\#}_{p,q}$ consist of all martingales for which

$$\|f\|_{H_{p,q}} := \|S(f)\|_{p,q} < \infty$$

and

$$\|f\|_{H^{\#}_{p,q}} := \|f^{\#}\|_{p,q} < \infty,$$

respectively. In case $p=q$ the usual definitions of Hardy spaces $H_{p,p}=H_p$ and $H^{\#}_{p,p}=H^{\#}_p$ are obtained. Note that it is unknown whether $H_{p,q}$ can be defined with f^*. We verified in [27] that

$$H_{p,q} \sim L_{p,q} \qquad (1<p<\infty, 0<q\leqslant\infty).$$

Recall that $L \log L \subset H^{\#}_1$, more exactly,

$$E(f^{\#}) \leqslant C + CE(|f| \log^+ |f|)$$

where $\log^+ u = 1_{\{u>1\}} \log u$ (see Garsia [13]).

The following interpolation result concerning Hardy–Lorentz spaces will be used several times in this paper (see Weisz [26] and [27]).

THEOREM A. *If a sublinear operator V is bounded from H_{p_0} to L_{p_0} and from H_{p_1} to L_{p_1} then it is also bounded from $H_{p,q}$ to $L_{p,q}$ if $p_0 < p < p_1$ and* $0 < q \leqslant \infty$.

3. QUASI-LOCAL OPERATORS

The atomic decomposition of the Hardy spaces in the two-parameter case is much more complicated than in the one-parameter case. One reason for this is that the support of a two-parameter atom is not a dyadic interval or square but an open set. This was proved in Bernard [3] and Weisz [27]. However, we now give a finer atomic decomposition and decompose the atoms into "elementary (rectangle) particles."

First of all we introduce some notations. Suppose $F \subset [0, 1)^2$ is open with respect to the topology induced by the dyadic rectangles, which means F is the union of countably many dyadic rectangles. Denote by $\mathscr{M}(F)$ the maximal dyadic subrectangles of F. Let $\mathscr{M}_1(F)$ denote those dyadic subrectangles $R \subset F$, $R = I \times J$ that are maximal in the x direction. In other words, if $S = I' \times J \supset R$ is a dyadic subrectangle of F then $I = I'$. Define $\mathscr{M}_2(F)$ similarly.

A function $a \in L_2$ is an H_p-atom if

(i) $\operatorname{supp} a \subset F$ for an open set $F \subset [0, 1)^2$

(ii) $\|a\|_2 \leqslant \lambda(F)^{1/2 - 1/p}$

(iii) a can be further decomposed into the sum of "elementary particles" $a_R \in L_2$ $(R \in \mathscr{M}(F))$ in the sense of

$$E_{n,m} a = \sum_{R \in \mathscr{M}(F)} E_{n,m} a_R \qquad \text{a.e.} \quad \text{for all} \quad n, m \in \mathbf{N},$$

satisfying

(α) $\operatorname{supp} a_R \subset R \subset F$

(β) for all $x, y \in [0, 1)$ and $R \in \mathscr{M}(F)$,

$$\int_0^1 a_R(x, y)\, dx = \int_0^1 a_R(x, y)\, dy = 0$$

(γ) $\displaystyle\left(\sum_{R \in \mathscr{M}(F)} \|a_R\|_2^2 \right)^{1/2} \leqslant \lambda(F)^{1/2 - 1/p}.$

If $a \in L_2$ satisfies (i) with a dyadic rectangle F, (ii), and (β) then a is called a *rectangle H_p-atom.*

Now the atomic decomposition of H_p is formulated.

THEOREM 1. *A martingale* $f=(f_{n,m}; n, m \in \mathbf{N})$ *is in* H_p $(0<p\leqslant 1)$ *if and only if there exist a sequence* $(a^k, k\in\mathbf{N})$ *of* H_p*-atoms and a sequence* $(\mu_k, k\in\mathbf{N})$ *of real numbers such that*

$$\begin{aligned} \sum_{k=0}^{\infty} \mu_k E_{n,m} a^k &= f_{n,m} \qquad \text{for all} \quad n, m \in \mathbf{N} \\ \sum_{k=0}^{\infty} |\mu_k|^p &< \infty. \end{aligned} \tag{6}$$

Moreover, the following equivalence of norms holds:

$$\|f\|_{H_p} \sim \inf \left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p} \tag{7}$$

where the infimum is taken over all decompositions of f of the form (6).

Proof. It is proved in Weisz [27] that there exist functions $a^k \in L_2$ satisfying (i) and (ii) and real numbers μ_k $(k\in\mathbf{N})$ such that (6) and one side of (7), more exactly, the inequality

$$\left(\sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p} \leqslant C_p \|f\|_{H_p},$$

hold.

Denote one of the functions a^k by a. Let F be the support of a. It is also verified in Weisz [27] that there exists a non-decreasing sequence $(F_{n,m}; n, m\in\mathbf{N})$ of sets, which means $F_{k,l}\subset F_{n,m}$ if $k\leqslant n$ and $l\leqslant m$, such that

$$F_{n,m} \in \mathscr{F}_{n-1,m-1} \qquad \text{and} \qquad \bigcup_{n,m\in\mathbf{N}} F_{n,m} = F.$$

Moreover,

$$a = \sum_{n,m\in\mathbf{N}} 1_{F_{n,m}} d_{n,m} a.$$

Equation (1) implies that the martingale difference are orthogonal, so we have

$$\|a\|_2^2 = E\left(\sum_{n,m\in\mathbf{N}} 1_{F_{n,m}} |d_{n,m} a|^2 \right). \tag{8}$$

Since $F_{n,m} \in \mathcal{F}_{n-1,m-1}$, it can be decomposed into a finite union of dyadic rectangles $F^k_{n,m}$, i.e.,

$$F_{n,m} = \bigcup_k F^k_{n,m}$$

with $F^k_{n,m} \in \mathcal{F}_{n-1,m-1}$.

To each $F^k_{n,m}$ we associate a maximal dyadic subrectangle $\hat{F}^k_{n,m}$ of F, i.e., $\hat{F}^k_{n,m} \in \mathcal{M}(F)$, such that $F^k_{n,m} \subset \hat{F}^k_{n,m}$. For $R \in \mathcal{M}(F)$ let

$$a_R := \sum_{n,m \in \mathbf{N}} \sum_{k : \hat{F}^k_{n,m} = R} 1_{F^k_{n,m}} d_{n,m} a.$$

It is easy to see that this sum converges a.e. and also in L_2 norm. Obviously,

$$E_{n,m} a = \sum_{R \in \mathcal{M}(F)} E_{n,m} a_R \qquad (n, m \in \mathbf{N})$$

since the sum of the right-hand side is finite for each $(n, m) \in \mathbf{N}^2$. Note that

$$a = \sum_{R \in \mathcal{M}(F)} a_R \qquad \text{in} \quad L_2$$

because of the orthogonality of the martingale differences. Since $F^k_{n,m} \subset R$, (α) is obvious. If $R \in \mathcal{F}_{N-1,M-1}$ then (1) implies that

$$E_{k,l} a_R = 0 \qquad \text{for all} \quad (k, l) \ngeq (N, M).$$

Henceforth

$$E_{k,\infty} a_R = 0 \qquad \text{for all} \quad k \leqslant N-1.$$

This yields that

$$\int_0^1 a_R(x, y)\, dx = 0.$$

The other equation of (β) can be proved in the same way. Using the orthogonality of the martingale difference and the fact that the sets

$F^k_{n,m}$ are disjoint for each fixed $n, m \in \mathbf{N}$ and (8), we can conclude that

$$\sum_{R \in \mathscr{M}(F)} \|a_R\|_2^2 = \sum_{R \in \mathscr{M}(F)} E\left(\sum_{n, m \in \mathbf{N}} \sum_{k : \hat{F}^k_{n,m} = R} 1_{F^k_{n,m}} |d_{n,m} a|^2 \right)$$
$$= E\left(\sum_{n, m \in \mathbf{N}} \sum_k 1_{F^k_{n,m}} |d_{n,m} a|^2 \right)$$
$$= \|a\|_2^2 \leqslant \lambda(F)^{1-2/p}$$

which proves (γ).

For the other side of (7) we prove that if a is an H_p-atom then

$$\|a\|_{H_p} \leqslant 1 \qquad (0 < p \leqslant 1).$$

Indeed, from the definition of the atom it follows that

$$d_{n,m} a = \sum_{R \in \mathscr{M}(F)} d_{n,m} a_R \qquad (n, m \in \mathbf{N})$$

and

$$\operatorname{supp} d_{n,m} a_R \subset R.$$

Hence

$$\operatorname{supp} S(a) \subset F.$$

Applying (2), (3), and Hölder's inequality we have

$$E(S^p(a)) \leqslant [E(S^2(a))]^{p/2} \lambda(F)^{1-p/2} \leqslant 1.$$

Assume that $0 < p \leqslant 1$ and f has a decomposition of the form (6). It is easy to check that in this case

$$S(f) \leqslant \sum_{k=0}^{\infty} |\mu_k| \, S(a^k). \tag{9}$$

Consequently,

$$E[S^p(f)] \leqslant \sum_{k=0}^{\infty} |\mu_k|^p \, E[S^p(a^k)] \leqslant \sum_{k=0}^{\infty} |\mu_k|^p \tag{10}$$

holds, which finishes the proof of the theorem. ∎

Note that H_p cannot be decomposed into rectangle H_p-atoms; a counterexample can be found in Weisz [27].

The analogue of this theorem in the classical case was shown by Fefferman [8].

Motivated by the definition in Móricz et al. [18] and Fefferman [8] we introduce the H_p-quasi-local operators. For each daydic interval I let I^r ($r \in \mathbf{N}$) be the dyadic interval for which $I \subset I^r$ and

$$\lambda(I^r) = 2^r \lambda(I).$$

If $R := I \times J$ is a dyadic rectangle then set

$$R^r := I^r \times J^r.$$

Although H_p cannot be decomposed into rectangle atoms, in the definition of quasi-local operators it is enough to take these atoms.

An operator V, which maps the set of martingales into the collection of measurable functions, is called *H_p-quasi-local* if there exists $\delta > 0$ such that for every rectangle H_p-atom a supported on the dyadic rectangle R and for every $r \geqslant 1$ one has

$$\int_{[0,1)^2 \backslash R^r} |Va|^p \, d\lambda \leqslant C_p 2^{-\delta r}.$$

Before stating the main result of this section we recall Journé's covering lemma in one of its forms. Opposed to the one-parameter case, an open subset of $[0,1)^2$ cannot be decomposed into disjoint maximal dyadic rectangles, however the following lemma holds.

LEMMA 1 (Journé [15]). *Assume that F is an open subset of $[0,1)^2$ and $R = I \times J$ belongs to $\mathcal{M}_2(F)$. Let $F_1 := \{(1_F)^* > 1/2\}$ and $\hat{I}$ be the maximal dyadic interval containing I such that $\hat{I} \times J \subset F_1$, i.e., $\hat{I} \times J \in \mathcal{M}_1(F_1)$. Set*

$$\gamma_1(R, F) := \frac{\lambda(\hat{I})}{\lambda(I)}.$$

Define $\gamma_2(R, F)$ similarly. Then

$$\lambda\left(\bigcup_{R \in \mathcal{M}_2(F)} \hat{I} \times J\right) \leqslant C\lambda(F) \tag{11}$$

and

$$\sum_{R \in \mathcal{M}_2(F)} \gamma_1(R, F)^{-\delta} \lambda(R) \leqslant C_\delta \lambda(F), \tag{12}$$

for every $\delta > 0$ *where* C_δ *depends only on* δ, *not on* F.

Of course there is a symmetric form of this lemma for rectangles in $\mathcal{M}_1(F)$.

Note that (11) follows easily from Markov's inequality and from (3):

$$\lambda(F_1) \leqslant 4E[(1_F)^{*2}] \leqslant CE(1_F^2) = C\lambda(F). \tag{13}$$

Journé's lemma is extended to higher dimensions by Pipher [20].

The following result says that for an operator V to be bounded from H_p to L_p $(0 < p \leqslant 1)$ it is enough to check V on rectangle H_p-atoms and the boundedness of V on L_2.

THEOREM 2. *Suppose that the operator* V *is sublinear and* H_p-*quasi-local for some* $0 < p \leqslant 1$. *If* V *is bounded from* L_2 *to* L_2 *then*

$$\|Vf\|_p \leqslant C_p \|f\|_{H_p} \qquad (f \in H_p).$$

Proof. Similarly to (9) and (10) it is enough to show that if a is an H_p-atom then

$$\|Va\|_p \leqslant C_p. \tag{14}$$

Let a be an H_p-atom with support F. Set

$$F_1 := \{(1_F)^* > 1/2\} \qquad \text{and} \qquad F_2 := \{(1_{F_1})^* > 1/2\}.$$

As in (13), we have

$$\lambda(F_2) \leqslant C\lambda(F_1) \leqslant C\lambda(F).$$

Given a dyadic rectangle $R = I \times J \in \mathcal{M}(F)$ define the dyadic interval $\hat{I}$ such that

$$\hat{I} \supset I \qquad \text{and} \qquad R' := \hat{I} \times J \in \mathcal{M}_1(F_1).$$

Furthermore define the dyadic interval $\hat{J}$ such that

$$\hat{J} \supset J \qquad \text{and} \qquad \hat{R} := \hat{I} \times \hat{J} \in \mathcal{M}_2(F_2).$$

Set

$$2^{r_1} := \gamma_1(R, F) := \frac{\lambda(\hat{I})}{\lambda(I)} \qquad \text{and} \qquad 2^{r_2} := \gamma_2(R', F_1) := \frac{\lambda(\hat{J})}{\lambda(J)}.$$

Take the decomposition

$$a = \sum_{R \in \mathcal{M}(F)} a_R$$

as in Theorem 1. Then

$$\int_{\bigcup_{R \in \mathcal{M}(F)} \hat{R}} |Va|^p \, d\lambda \leqslant \lambda\left(\bigcup_{R \in \mathcal{M}(F)} \hat{R}\right)^{1-p/2} \left(\int_{[0,1)^2} |Va|^2 \, d\lambda\right)^{p/2}$$

$$\leqslant \lambda(F_2)^{1-p/2} \lambda(F)^{p/2-1} \leqslant C_p.$$

So we have to consider

$$\int_{[0,1)^2 \setminus \bigcup_{R \in \mathcal{M}(F)} \hat{R}} |Va|^p \, d\lambda \leqslant \sum_{R \in \mathcal{M}(F)} \int_{[0,1)^2 \setminus \hat{R}} |Va_R|^p \, d\lambda.$$

Obviously,

$$\int_{[0,1)^2 \setminus \hat{R}} |Va_R|^p \, d\lambda \leqslant \int_{([0,1) \setminus \hat{I}) \times [0,1)} |Va_R|^p \, d\lambda + \int_{[0,1) \times ([0,1) \setminus \hat{J})} |Va_R|^p \, d\lambda.$$

Observe that

$$\int_{([0,1) \setminus \hat{I}) \times [0,1)} |Va_R|^p \, d\lambda \leqslant \int_{[0,1)^2 \setminus (\hat{I} \times J^{r_1})} |Va_R|^p \, d\lambda = \int_{[0,1)^2 \setminus R^{r_1}} |Va_R|^p \, d\lambda.$$

Since

$$\frac{a_R}{\|a_R\|_2} \lambda(R)^{1/2-1/p}$$

is a rectangle H_p-atom, we have by the H_p-quasi-locality that

$$\int_{([0,1) \setminus \hat{I}) \times [0,1)} |Va_R|^p \, d\lambda \leqslant C_p 2^{-\delta r_1} \|a_R\|_2^p \, \lambda(R)^{1-p/2}$$

$$= C_p \gamma_1(R, F)^{-\delta} \|a_R\|_2^p \, \lambda(R)^{1-p/2}.$$

By Hölder's inequality and Journé's lemma,

$$\sum_{R \in \mathcal{M}(F)} \int_{([0,1)\setminus \hat{I}) \times [0,1)} |Va_R|^p \, d\lambda$$
$$\leqslant C_p \left(\sum_{R \in \mathcal{M}(F)} \|a_R\|_2^2 \right)^{p/2} \left(\sum_{R \in \mathcal{M}(F)} \gamma_1(R, F)^{-2\delta/(2-p)} \lambda(R) \right)^{1-p/2}$$
$$\leqslant C_p \lambda(F)^{p/2-1} \lambda(F)^{1-p/2} = C_p.$$

Similarly,

$$\sum_{R \in \mathcal{M}(F)} \int_{[0,1) \times ([0,1)\setminus \hat{J})} |Va_R|^p \, d\lambda$$
$$\leqslant C_p \sum_{R \in \mathcal{M}(F)} \gamma_2(R', F_1)^{-\delta} \|a_R\|_2^p \lambda(R)^{1-p/2}$$
$$\leqslant C_p \lambda(F)^{p/2-1} \left(\sum_{R \in \mathcal{M}(F)} \gamma_2(R', F_1)^{-2\delta/(2-p)} \lambda(R) \right)^{1-p/2}.$$

It is easy to see that if $R_1, R_2 \in \mathcal{M}(F)$ and $R_1' = R_2'$ then $R_1 \cap R_2 = \emptyset$ or $R_1 = R_2$. Recall that $R' \in \mathcal{M}_1(F_1)$. So

$$\sum_{R \in \mathcal{M}(F)} \gamma_2(R', F_1)^{-2\delta/(2-p)} \lambda(R) = \sum_{S \in \mathcal{M}_1(F_1)} \left(\sum_{R' = s} \lambda(R) \right) \gamma_2(S, F_1)^{-2\delta/(2-p)}$$
$$\leqslant \sum_{S \in \mathcal{M}_1(F_1)} \lambda(S)\, \gamma_2(S, F_1)^{-2\delta/(2-p)}$$
$$\leqslant C_p \lambda(F_1) \leqslant C_p \lambda(F)$$

where we applied again Journé's lemma and (13). Consequently,

$$\sum_{R \in \mathcal{M}(F)} \int_{[0,1) \times ([0,1)\setminus \hat{J})} |Va_R|^p \, d\lambda \leqslant C_p$$

which proves (14) as well as the theorem. ∎

This result in the classical case is due to Fefferman [8].

Theorem A and (5) imply

COROLLARY 1. *Suppose that the sublinear operator V is H_p-quasi-local for each $p_0 < p \leqslant 1$ ($p_0 < 1$). If V is bounded from L_2 to L_2 then*

$$\|Vf\|_{p,q} \leqslant C_{p,q} \|f\|_{H_{p,q}} \qquad (f \in H_{p,q})$$

for every $p_0 < p < 2$ *and* $0 < q \leqslant \infty$. *Specially,* V *is of weak type* $(H_1^{\#}, L_1)$, *i.e., if* $f \in H_1^{\#}$ *then*

$$\|Vf\|_{1,\infty} = \sup_{\alpha > 0} \alpha\lambda(|Vf| > \alpha) \leqslant C_1 \|f\|_{H_{1,\infty}}$$

$$= C_1 \sup_{\alpha > 0} \alpha\lambda(S(f) > \alpha) \leqslant C_1 \|f\|_{H_1^{\#}}.$$

4. CESÀRO SUMMABILITY OF DOUBLE WALSH–FOURIER SERIES

First the Walsh system is to be introduced. Every point $x \in [0, 1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}, \qquad 0 \leqslant x_k < 2, \; x_k \in \mathbf{N}.$$

In case there are two different forms, we choose the one for which $\lim_{k \to \infty} x_k = 0$.

The functions

$$r_n(x) := \exp(\pi x_n \sqrt{-1}) \qquad (n \in \mathbf{N})$$

are called *Rademacher functions.*

The product system generated by these functions is the *one-dimensional Walsh system*

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k}$$

where $n \in \sum_{k=0}^{\infty} n_k 2^k$, $0 \leqslant n_k < 2$, and $n_k \in \mathbf{N}$.

The Kronecker product $(w_{n,m}; n, m \in \mathbf{N})$ of two Walsh systems is said to be the *two-dimensional Walsh system*. Thus

$$w_{n,m}(x, y) := w_n(x)\, w_m(y).$$

Recall that the *Walsh–Dirichlet kernels*

$$D_n := \sum_{k=0}^{n-1} w_k$$

satisfy

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if} \quad x \in [0, 2^{-n}) \\ 0 & \text{if} \quad x \in [2^{-n}, 1) \end{cases} \tag{15}$$

for $n \in \mathbf{N}$ (see Fine [10]).

If $f \in L_1$ then the number

$$\hat{f}(n, m) := E(f w_{n, m})$$

is said to be the (n, m)th *Walsh–Fourier coefficient* of f $(n, m \in \mathbf{N})$. Let us extend this definition to martingales as well. If $f = (f_{k, l}; k, l \in \mathbf{N})$ is a martingale then let

$$\hat{f}(n, m) := \lim_{\min(k, l) \to \infty} E(f_{k, l} w_{n, m}) \qquad (n, m \in \mathbf{N}).$$

Since $w_{n, m}$ is $\mathcal{F}_{k, k}$ measurable for $n, m < 2^k$, it can immediately be seen that this limit does exist. Note that if $f \in L_1$ then $E_{k, l} f \to f$ in L_1 norm as $k, l \to \infty$, hence

$$\hat{f}(n, m) = \lim_{\min(k, l) \to \infty} E((E_{k, l} f)\, w_{n, m}) \qquad (n, m \in \mathbf{N}).$$

Thus the Walsh–Fourier coefficients of $f \in L_1$ are the same as the ones of the martingale $(E_{k, l} f; k, l \in \mathbf{N})$ obtained from f.

Denote by $s_{n, m} f$ the (n, m)th partial sum of the Walsh–Fourier series of a martingale f, namely,

$$s_{n, m} f := \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \hat{f}(k, l)\, w_{k, l}.$$

It is easy to see that

$$s_{2^n, 2^m} f = f_{n, m}. \tag{16}$$

Recall that the *Walsh–Fejér kernels*

$$K_n := \frac{1}{n} \sum_{k=1}^{n} D_n \qquad (n \in \mathbf{N})$$

satisfy

$$|K_n(x)| \leqslant \sum_{j=0}^{N-1} 2^{j-N} \sum_{i=j}^{N-1} (D_{2^i}(x) + D_{2^i}(x \dotplus 2^{-j-1})) \tag{17}$$

for $x \in [0, 1)$, n, $N \in \mathbf{N}$, and $2^{N-1} \leqslant n < 2^N$, and

$$K_{2^n}(x) = \frac{1}{2}\left(2^{-n} D_{2^n}(x) + \sum_{j=0}^{n} 2^{j-n} D_{2^n}(x \dotplus 2^{-j-1})\right) \tag{18}$$

for $x \in [0, 1)$ and $n \in \mathbf{N}$, where $\dotplus$ denotes the dyadic addition (see e.g. Shipp *et al.* [23]). For $n, m \in \mathbf{N}$ and a martingale f the *Cesàro mean* of order (n, m) of the double Walsh–Fourier series of f is given by

$$\sigma_{n,m} f := \frac{1}{nm} \sum_{k=1}^{n} \sum_{l=1}^{m} s_{k,l} f.$$

It is simple to show that

$$\sigma_{n,m} f(x, y) = \int_0^1 \int_0^1 f(t, u)\, K_n(x \dotplus t)\, K_m(y \dotplus u)\, dt\, du$$

if $f \in L_1$.

For a martingale f we consider the maximal operators

$$\sigma^* f := \sup_{n, m \in \mathbf{N}} |\sigma_{n,m} f|, \qquad \sigma f := \sup_{n, m \in \mathbf{N}} |\sigma_{2^n, 2^m} f|.$$

It is shown in Weisz [28] that

$$\|\sigma^* f\|_p \leqslant C_p \|f\|_p \qquad (1 < p \leqslant \infty). \tag{19}$$

Now we state our main result.

Theorem 3. *There are absolute constants C and $C_{p,q}$ such that*

$$\|\sigma^* f\|_{p,q} \leqslant C_{p,q} \|f\|_{H_{p,q}} \qquad (f \in H_{p,q}) \tag{20}$$

for every $4/5 < p < \infty$ and $0 < q \leqslant \infty$. Especially, if $f \in H_1^{\#}$ then

$$\lambda(\sigma^* f > \alpha) \leqslant \frac{C}{\alpha} \|f^{\#}\|_1 \qquad (\alpha > 0). \tag{21}$$

Proof. By Corollary 1, (19), and Wolff's interpolation theorem [29] (see also Weisz [27]), the proof of Theorem 3 will be complete if we show that the operator σ^* is H_p-quasi-local for every $4/5 < p \leqslant 1$.

Let a be a rectangle H_p-atom with support $R = I \times J$ and $\lambda(I) = 2^{-K}$, $\lambda(J) = 2^{-L}$ $(K, L \in \mathbf{N})$. Without loss of generality we can suppose that

$I=[0, 2^{-K})$ and $J=[0, 2^{-L})$. It is easy to see that $\hat{a}(n, m)=0$ if $n<2^K$ or $m<2^L$, so, in this case, $\sigma_{n,m}a=0$. Therefore we assume that $n\geqslant 2^K$ and $m\geqslant 2^L$.

To prove the quasi-locality of σ^* we have to integrate $|\sigma^* a|^p$ over $[0, 1)^2\backslash R^r$ where $r\geqslant 1$ is arbitrary. We do this in three steps.

Step 1: *Integrating over* $([0, 1)\backslash I^r)\times J^r$. If $j\geqslant K-r$ and $x\notin I^r$ then $x \dotplus 2^{-j-1}\notin I^r$. Consequently, for $x\notin I^r$ and $i\geqslant j\geqslant K-r$ we have

$$a(t, u)\, D_{2^i}(x \dotplus t)=a(t, u)\, D_{2^i}(x \dotplus t \dotplus 2^{-j-1})=0.$$

Recall that $2^N>n\geqslant 2^{N-1}$, so $N-1\geqslant K$. Henceforth, for $x\notin I^r$,

$$\begin{aligned}
|\sigma_{n,m}a(x, y)| &\leqslant \int_I \left| \int_J a(t, u)\, K_m(y \dotplus u)\, du \right| |K_n(x \dotplus t)|\, dt \\
&\leqslant \sum_{j=0}^{N-1} 2^{j-N} \sum_{i=j}^{N-1} \int_I \left| \int_J a(t, u)\, K_m(y \dotplus u)\, du \right| \\
&\qquad (D_{2^i}(x \dotplus t)+D_{2^i}(x \dotplus t \dotplus 2^{-j-1}))\, dt \\
&\leqslant C2^{-K} \sum_{j=0}^{K-r-1} 2^j \sum_{i=j}^{K-1} \int_I \left| \int_J a(t, u)\, K_m(y \dotplus u)\, du \right| \\
&\qquad (D_{2^i}(x \dotplus t)+D_{2^i}(x \dotplus t \dotplus 2^{-j-1}))\, dt \\
&\quad + C \sum_{j=0}^{K-r-1} 2^j \sum_{i=K}^{\infty} 2^{-i} \int_I \left| \int_J a(t, u)\, K_m(y \dotplus u)\, du \right| \\
&\qquad (D_{2^i}(x \dotplus t)+D_{2^i}(x \dotplus t \dotplus 2^{-j-1}))\, dt.
\end{aligned}$$

Observe that the right hand side is independent of n. Therefore

$$\begin{aligned}
\sigma^* a(x, y) &\leqslant C2^{-K} \sum_{j=0}^{K-r-1} 2^j \sum_{i=j}^{K-1} \int_I (D_{2^i}(x \dotplus t)+D_{2^i}(x \pm t \dotplus 2^{-j-1})) \\
&\qquad \sup_{m\in\mathbf{N}} \left| \int_J a(t, u)\, K_m(y \dotplus u)\, du \right| dt \\
&\quad + C \sum_{j=0}^{K-r-1} 2^j \sum_{i=K}^{\infty} 2^{-i} \int_I (D_{2^i}(x \dotplus t)+D_{2^i}(x \dotplus t \dotplus 2^{-j-1})) \\
&\qquad \sup_{m\in\mathbf{N}} \left| \int_J a(t, u)\, K_m(y \dotplus u)\, du \right| dt \\
&=: (A)+(B).
\end{aligned}$$

By Hölder's inequality,

$$\int_{J^r} (A)^p \, dy$$

$$\leqslant C_p \lambda(J^r)^{1-p} \, 2^{-Kp} \sum_{j=0}^{K-r-1} 2^{jp} \sum_{i=j}^{K-1} \bigg(\int_I (D_{2^i}(x \dot{+} t) + D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})) \int_{J^r} \sup_{m \in \mathbf{N}} \left| \int_J a(t,u) \, K_m(y \dot{+} u) \, du \right| dy \, dt \bigg)^p.$$

Using again Hölder inequality and (19) for one dimension and for a fixed t, we obtain

$$\int_{J^r} \sup_{m \in \mathbf{N}} \left| \int_J a(t,u) \, K_m(y \dot{+} u) \, du \right| dy \leqslant \lambda(J^r)^{1/2} \left(\int_0^1 \sup_{m \in \mathbf{N}} \left| \int_J a(t,u) \, K_m(y \dot{+} u) \, du \right|^2 dy \right)^{1/2} \leqslant \lambda(J^r)^{1/2} \left(\int_0^1 |a(t,y)|^2 \, dy \right)^{1/2}.$$

Hence

$$\int_{J^r} (A)^p \, dy \leqslant C_p \lambda(J^r)^{1-p/2} \, 2^{-Kp} \sum_{j=0}^{K-r-1} 2^{jp} \sum_{i=j}^{K-1} \bigg(\int_I \left(\int_0^1 |a(t,y)|^2 \, dy \right)^{1/2} \times (D_{2^i}(x \dot{+} t) + D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1})) \, dt \bigg)^p$$

Observe by (15) that

$$1_{[0, 2^{-K})}(t) \, D_{2^i}(x \dot{+} t) = 2^i 1_{[2^{-K+r}, 2^{-i})}(x) \, 1_{[0, 2^{-K})}(t)$$

and

$$1_{[0, 2^{-K})}(t) \, D_{2^i}(x \dot{+} t \dot{+} 2^{-j-1}) = 2^i 1_{[2^{-j-1}, 2^{-j-1} + 2^{-i})}(x) \, 1_{[0, 2^{-K})}(t)$$

if $j \leqslant i \leqslant K-1$ and $x \notin [0, 2^{-K+r})$. So

$$\int_{J^r} (A)^p \, dy \leqslant C_p \lambda(J^r)^{1-p/2} \, 2^{-Kp} \sum_{j=0}^{K-r-1} 2^{jp} \sum_{i=j}^{K-1} 2^{ip} \left(\int_I \left(\int_0^1 |a(t,y)|^2 \, dy \right)^{1/2} dt \right)^p \times [1_{[2^{-K+r}, 2^{-i})}(x) + 1_{[2^{-j-1}, 2^{-j-1} + 2^{-i})}(x)].$$

Using Hölder's inequality and the definition of the atom we get that

$$\int_I \left(\int_0^1 |a(t, y)|^2 \, dy \right)^{1/2} dt \leqslant \left(\int_0^1 \int_0^1 |a(t, y)|^2 \, dy \, dt \right)^{1/2} 2^{-K/2}$$
$$\leqslant 2^{-K/2 + K/p - L/2 + L/p} 2^{-K/2}.$$

Hence

$$\int_{[0, 1) \setminus I^r} \int_{J^r} (A)^p \, dx \, dy \leqslant C_p 2^{r(1-p/2)} 2^{-2Kp+K} \sum_{j=0}^{K-r-1} 2^{jp} \sum_{i=j}^{K-1} 2^{i(p-1)}$$
$$\leqslant C_p 2^{r(1-p/2)} 2^{-2Kp+K} \sum_{j=0}^{K-r-1} 2^{j(2p-1)}$$
$$\leqslant C_p 2^{r(2-5p/2)}$$

if $p < 1$. For $4/5 < p < 1$ let $\delta := 5p/2 - 2 > 0$. If $p = 1$ then

$$\int_{[0, 1) \setminus I^r} \int_{J^r} (A)^p \, dx \, dy \leqslant C 2^{r/2} \sum_{j=0}^{K-r-1} 2^{j-K}(K-j) \leqslant C 2^{r/2} \sum_{k=r+1}^{\infty} k 2^{-k}.$$

Since

$$\sum_{k=r+1}^{\infty} k 2^{-k} = 2^{-r}(r+2) \leqslant C_\gamma 2^{-r\gamma}$$

for an arbitrary $0 < \gamma < 1$, we have

$$\int_{[0, 1) \setminus I^r} \int_{J^r} (A)^p \, dx \, dy \leqslant C 2^{-r\delta}$$

with $\delta := \gamma - 1/2$. If we choose $1/2 < \gamma < 1$ then $\delta > 0$.

Similarly,

$$\int_{J^r} (B)^p \, dy \leqslant C_p \lambda(J^r)^{1-p/2} \sum_{j=0}^{K-r-1} 2^{jp} \sum_{i=K}^{\infty} 2^{-ip} \left(\int_I \left(\int_0^1 |a(t, y)|^2 \, dy \right)^{1/2} \right.$$
$$\left. \times (D_{2^i}(x \dotplus t) + D_{2^i}(x \dotplus t \dotplus 2^{-j-1})) \, dt \right)^p.$$

If $i \geqslant K$ and $x \notin [0, 2^{-K+r})$ then

$$1_{[0, 2^{-K})}(t) \, D_{2^i}(x \dotplus t) = 0$$

and

$$1_{[0,2^{-K})}(t)\, D_{2^i}(x \dotplus t \dotplus 2^{-j-1}) = 2^i \sum_{k=1}^{2^{-K+i}} 1_{[2^{-j-1}+(k-1)2^{-i},\, 2^{-j-1}+k2^{-i})}(x)\, 1_{[(k-1)2^{-i},\, k2^{-i})}(t).$$

Therefore

$$\int_{J^r} (B)^p \, dy \leqslant C_p \lambda(J^r)^{1-p/2} \sum_{j=0}^{K-r-1} 2^{jp} \sum_{i=K}^{\infty} 2^{-ip} 2^{ip} \sum_{k=1}^{2^{-K+i}} 1_{[2^{-j-1}+(k-1)2^{-i},\, 2^{-j-1}+k2^{-i})}(x) \times \left(\int_I \left(\int_0^1 |a(t,y)|^2 \, dy \right)^{1/2} 1_{[(k-1)2^{-i},\, k2^{-i})}(t)\, dt \right)^p.$$

Using the inequality

$$\int_I \left(\int_0^1 |a(t,y)|^2 \, dy \right)^{1/2} 1_{[(k-1)2^{-i},\, k2^{-i})}(t)\, dt \leqslant 2^{-K/2+K/p-L/2+L/p} 2^{-i/2}$$

we conclude

$$\int_{[0,1)\backslash I^r} \int_{J^r} (B)^p \, dx\, dy \leqslant C_p 2^{r(1-p/2)} 2^{-Kp/2} \sum_{j=0}^{K-r-1} 2^{jp} \sum_{i=K}^{\infty} 2^{-ip/2} \leqslant C_p 2^{r(1-3p/2)}.$$

Consequently,

$$\int_{[0,1)\backslash I^r} \int_{J^r} |\sigma^* a(x,y)|^p \, dx\, dy \leqslant C_p 2^{-r\delta} \tag{22}$$

with $\delta := 5p/2 - 2 > 0$ if $4/5 < p < 1$ and $\delta := \gamma - 1/2$ $(1/2 < \gamma < 1)$ if $p = 1$.

Step 2: *Integrating over* $([0,1)\backslash I^r) \times [(0,1)\backslash J^r)$. Similarly to Step 1 we get for $x \notin I^r$ and $y \notin J^r$ that

$$
\begin{aligned}
&|\sigma_{n,m}a(x,y)| \\
&\leqslant \sum_{j=0}^{K-r-1} 2^{j-N} \sum_{i=j}^{N-1} \sum_{k=0}^{L-r-1} 2^{k-M} \sum_{l=k}^{M-1} \alpha \\
&\leqslant C2^{-K} \sum_{j=0}^{K-r-1} 2^{j} \sum_{i=j}^{K-1} 2^{-L} \sum_{k=0}^{L-r-1} 2^{k} \sum_{l=k}^{L-1} \alpha \\
&\quad + C2^{-K} \sum_{j=0}^{K-r-1} 2^{j} \sum_{i=j}^{K-1} \sum_{k=0}^{L-r-1} 2^{k} \sum_{l=L}^{\infty} 2^{-l}\alpha \\
&\quad + C \sum_{j=0}^{K-r-1} 2^{j} \sum_{i=K}^{\infty} 2^{-i}2^{-L} \sum_{k=0}^{L-r-1} 2^{k} \sum_{l=k}^{L-1} \alpha \\
&\quad + C \sum_{j=0}^{K-r-1} 2^{j} \sum_{i=K}^{\infty} 2^{-i} \sum_{k=0}^{L-r-1} 2^{k} \sum_{l=L}^{\infty} 2^{-l}\alpha \\
&=: (A)+(B)+(C)+(D)
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha := \int_I \int_J |a(t,u)| \, &(D_{2^i}(x \dotplus t) + D_{2^i}(x \dotplus t \dotplus 2^{-j-1})) \\
&\times (D_{2^l}(y \dotplus u) + D_{2^l}(y \dotplus u \dotplus 2^{-k-1}))\, dt\, du.
\end{aligned}
$$

With the method used in Step 1 we van verify that

$$
\begin{aligned}
\int_{[0,1)\backslash I^r} \int_{[0,1)\backslash J^r} (A)^p \, dx\, dy \leqslant C_p 2^{-Kp} &\sum_{j=0}^{K-r-1} 2^{jp} \sum_{i=j}^{K-1} 2^{-Lp} \sum_{k=0}^{L-r-1} 2^{kp} \sum_{l=k}^{L-1} \\
&2^{ip}2^{lp}2^{-i}2^{-l}2^{-Kp/2-Lp/2+K+L}2^{-Kp/2-Lp/2} \\
&\leqslant C_p 2^{-r\delta}
\end{aligned} \tag{23}
$$

with $\delta := 4p-2$ if $4/5 < p < 1$ and $\delta := 2\gamma - 1 (1/2 < \gamma < 1)$ if $p = 1$ and

$$
\begin{aligned}
\int_{[0,1)\backslash I^r} \int_{[0,1)\backslash J^r} (B)^p \, dx\, dy \leqslant C_p 2^{-Kp} &\sum_{j=0}^{K-r-1} 2^{jp} \sum_{i=j}^{K-1} \sum_{k=0}^{L-r-1} 2^{kp} \sum_{l=L}^{\infty} 2^{-lp} \\
&2^{ip}2^{lp}2^{-i}2^{-L}2^{-Kp/2-Lp/2+K+L}2^{-Kp/2-Lp/2} \\
&\leqslant C_p 2^{-r\delta}
\end{aligned} \tag{24}
$$

with $\delta := 3p - 1$ if $4/5 < p < 1$ and $\delta := \gamma + 1/2$ if $p = 1$. The estimation of (C) is similar. Finally,

$$\int_{[0,1)\backslash I^r} \int_{[0,1)\backslash J^r} (D)^p \, dx \, dy \leqslant C_p \sum_{j=0}^{K-r-1} 2^{jp} \sum_{i=K}^{\infty} 2^{-ip} \sum_{k=0}^{L-r-1} 2^{kp} \sum_{l=L}^{\infty} 2^{-lp}$$

$$2^{ip} 2^{lp} 2^{-K} 2^{-L} 2^{-Kp/2 - Lp/2 + K + L} 2^{-ip/2 - lp/2}$$

$$\leqslant C_p 2^{-2rp}. \tag{25}$$

Step 3: *integrating over* $I^r \times ([0,1)\backslash J^r)$. This case is analogue to Step 1.

Combining (22)–(25) we can establish that

$$\int_{[0,1)^2 \backslash R^r} |\sigma^* a(x,y)|^p \, dx \, dy \leqslant C_p 2^{-r\delta}$$

where $\delta := 5p/2 - 2 > 0$ if $4/5 < p < 1$ and $\delta := \gamma - 1/2$ $(1/2 < \gamma < 1)$ if $p = 1$. The proof of the theorem is complete. ∎

Note that, in the one-parameter case, (20) was proved by Fujii [12] for $p = q = 1$ (see also Schipp and Simon [22]), (21) by Schipp [21], and, moreover, in the two-parameter case, (21) was shown by Móricz, *et al.* [18] for the $L \log L$ class instead of $H_1^{\#}$. This theorem concerning the maximal Cesàro operators restricted on a cone can be found in Weisz [25] (see also Móricz, et al. [18]).

By (16) it is easy to show that the two-dimensional Walsh polynomials are dense in $H_1^{\#}$. Hence (21) and the usual density argument (see Marcinkievicz and Zygmund [16]) imply

COROLLARY 2. *If* $f \in H_1^{\#}$ *then*

$$\sigma_{n,m} f \to f \qquad \text{a.e.} \quad \text{as} \quad \min(n,m) \to \infty.$$

Considering the operator σ and (18) one can extend Theorem 2 to every $2/3 < p < \infty$. The proof is similar to that of Theorem 3 and is left to the reader.

THEOREM 4. *There are absolute constants* $C_{p,q}$ *such that*

$$\|\sigma f\|_{p,q} \leqslant C_{p,q} \|f\|_{H_{p,q}} \qquad (f \in H_{p,q})$$

for every $2/3 < p < \infty$ *and* $0 < q \leqslant \infty$.

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